

The circle problem:

$\mathbb{Z}^2 = \{(m, n) \mid m, n \in \mathbb{Z}\}$ . Give a "nice" region  $\Omega \in \mathbb{R}^2$

$$N_\Omega(R) = \#\mathbb{Z}^2 \cap R\Omega = \#\{(m, n) \in \mathbb{Z}^2 \mid (\frac{m}{R}, \frac{n}{R}) \in \Omega\}$$

estimate  $N(R)$  as  $R \rightarrow \infty$ .

Let  $\Omega$  be an interval  $\Omega = (\alpha, \beta) \in \mathbb{R}$

$$N_\Omega(R) = \#\{n \in \mathbb{Z} \mid R\alpha < n < R\beta\}$$

Then!  $N_\Omega(R) = \text{length}(R(\beta - \alpha)) + O(1) = R \text{length}(\Omega) + O(1)$

For  $\Omega = \text{unit Disc}$ , this is the circle problem,

First guess:  $N_\Omega(R) \sim \text{area}(R\Omega) = R^2 \text{area}(\Omega)$

After we show this, then: the remainder:

$$P(R) = N_\Omega(R) - \text{area}(R\Omega), \text{ the correct size of } P(R).$$

Easy:  $P(R) \ll \text{length}(\partial(R\Omega))$

Will treat  $\Omega = B(0, 1)$  the circle/disk.

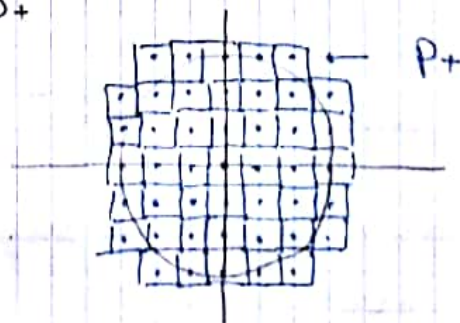
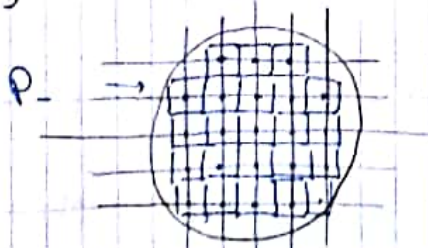
Proof 1:

Let  $P_- = \text{union of all unit squares centered at a lattice point } p \in \mathbb{Z}^2 \text{ s.t. } \square_p \subset B(0, R)$ .  $P_-$  is clearly a polygon,  $P_- \subset R\Omega$

$$\# P_- \cap \mathbb{Z}^2 = \text{area } P_- = \sum_{p \in P_-} \text{Area}(\square_p)$$

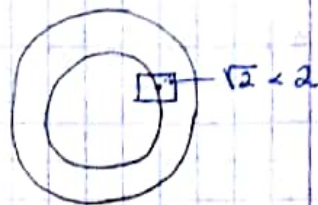
$P_+ = \text{union}_{p \in \mathbb{Z}^2} \square_p \text{ s.t. } \square_p \cap B(0, R) \neq \emptyset$ .  $P_+$  is a polygon,  $B(0, R) \subset P_+$   
 $\text{Area}(P_+) = \# P_+ \cap \mathbb{Z}^2$

We got:  $P_- \subset B(0, R) \subset P_+$



Claim:

$$B(0, R-2) \subseteq P_- \quad \& \quad P_+ \subseteq B(0, R+2)$$



Finally:

$$\text{Area } P_- = \#P_- \cap \mathbb{Z}^2 \leq N(R) \leq \#P_+ \cap \mathbb{Z}^2 = \text{area } P_+$$

We get:  $\text{Area}(B(R-2)) \leq N(R) \leq \text{Area}(B(R+2))$

$$\pi(R-2)^2 = \pi R^2 - 4\pi R + 4 \qquad \pi(R+2)^2 = \pi R^2 + 4\pi R + 4$$

$$\Rightarrow |N(R) - \pi R^2| \leq 4\pi R + 4\pi = O(R)$$

$$\text{So } N(R) = \pi R^2 + O(R)$$

Proof:

Will work for any convex region with piecewise smooth boundary. (Think about exact requirements). Clearly we need boundary to have measure 0.

$$\text{We found } N(R) = \text{Area}(R; \Omega) + O(\text{length}(\partial R; \Omega))$$

"Circle problem":

$$N(R) - \text{area}(R; \Omega) \ll (\text{length}(\partial R; \Omega))^{\frac{1}{2} + \epsilon} \quad \forall \epsilon > 0$$

Intuition:  $N(R) - \text{area} = \sum \pm \Delta$

Assume that  $\pm$  are "random" & the areas are on average  $\frac{1}{2}$ , & over/under estimations are independent. Then

$$\# \text{ of } \pm \Delta \approx \text{length}(\text{boundary})$$

If we forget  $\pm$  we get  $O(R)$ .

Central Limit Theorem:

$X_1, \dots, X_n$  IID random variables eg.  $X_i = \frac{1}{2} \epsilon_i$

$$\epsilon_i = \begin{cases} 1 & \text{Prob } \frac{1}{2} \\ -1 & \text{Prob } \frac{1}{2} \end{cases}$$

$S_n = X_1 + \dots + X_n - \frac{n}{2}$  will typically be of size  $\sqrt{n}$ .



$$\mathbb{E} X_i = 0 \quad S_n = \sum_{i=1}^n (X_i - \mathbb{E}(X_i)) \quad \mathbb{E}(S_n) = 0$$

$$\text{Var}(S_n) = \mathbb{E}(S_n^2) = c \cdot n$$

$$\text{Var}(X_i) = \mathbb{E}(|X_i - \mathbb{E}(X_i)|^2) = \frac{1}{2} |1/2| + \frac{1}{2} |-1/2| \quad \rho^2 = \frac{1}{2}$$

CZT:

$$\frac{S_n}{\sqrt{\text{Var}(S_n)}} \sim N(0,1) \quad \text{Prob} \left( a < \frac{S_n}{\sqrt{\text{Var}}} < b \right) \xrightarrow{n \rightarrow \infty} \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} > 0$$

i.e.  $\forall c > 0$ , with positive prob.  $|\frac{S_n}{\sqrt{n}}| > c$ , but 100% of  $\frac{S_n}{\sqrt{n}}$  is finite.

So we are led to expect that for most  $R$ 's

$$P(R) = N(R) - \text{area}(R, R) \text{ will be about } \sqrt{\text{length } \partial(R)}$$

& we could expect  $|P(R)|$  not to be much larger.

Goal:

1) Improve bound  $p(R) = o(R)$

2) Can we give argument to show that we cannot do better than  $o(R^{1/2})$

Cramer 1920:

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \left( \frac{P(r)}{r} \right)^2 dr = c > 0$$

$\Rightarrow$  for arbitrarily large  $R$ 's we have  $|P(R)| > * R^{1/2}$

i.e. " $P(R) = \Omega(R^{1/2})$ ".

Proof 2: (slicing & reduce to 1-dim Problem)

$$N(R) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \leq R}} \#(Z_n \cap \mathbb{Z}^2)$$

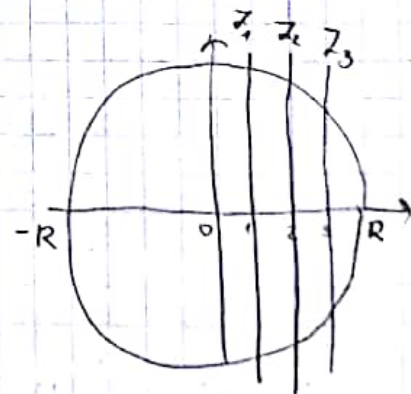
$$Z_n = \text{line } (n, t), \quad t \in \mathbb{R}, \quad n^2 + t^2 \leq R^2$$

$$\# Z_n = \text{length}(Z_n) + o(1) = 2\sqrt{R^2 - n^2} + o(1)$$

$$n^2 + t^2 \leq R^2 \Rightarrow t^2 \leq R^2 - n^2$$

By the one-dim estimate.

$$\begin{aligned} N(R) &= \sum_{|n| \leq R} \{ 2\sqrt{R^2 - n^2} + o(1) \} = 2 \cdot 2 \sum_{n=1}^R \sqrt{R^2 - n^2} + 2R + o(1) + o(R) = \\ &= 4 \sum_{n=1}^R \sqrt{R^2 - n^2} + o(R) \end{aligned}$$



$$\sum_{n=1}^R \sqrt{R^2 - n^2}$$

use summation by parts  $\sum_{0 < n \leq R} a_n f(n) = A(R)f(R) - \int_0^R A(t)f'(t) dt$ ,  $A(t) = \sum_{0 < n \leq t} a_n$

take  $f(t) = R^2 - t^2$ ,  $a_n = 1$ ,  $A(t) = \sum_{0 < n \leq t} 1 = \lfloor t \rfloor$ ,  $f' = \frac{-2t}{\sqrt{R^2 - t^2}}$

$$\sum_{1 \leq n \leq R} \sqrt{R^2 - n^2} = \underbrace{\lfloor R \rfloor \sqrt{R^2 - R^2}}_0 - \int_0^R \lfloor t \rfloor \frac{-2t}{\sqrt{R^2 - t^2}} dt =$$

$$= \int_0^R \frac{(t - \lfloor t \rfloor) 2t}{\sqrt{R^2 - t^2}} dt = \int_0^R \frac{t^2}{R^2 - t^2} dt + O\left(\int_0^R \frac{t}{\sqrt{R^2 - t^2}} dt\right)$$

$$\int_0^R \frac{t^2}{\sqrt{R^2 - t^2}} dt = \int_0^R \frac{R^2 u^2}{R(1 - u^2)} R du = R^2 \int_0^{\pi/4} \frac{u^2}{\sqrt{1 - u^2}} du = -\sqrt{R^2 - t^2} \Big|_0^R = R$$

$t = Ru$   
 $dt = R du$

We find:

$$N(R) = 4 \sum_{n=1}^{\lfloor R \rfloor} \sqrt{R^2 - n^2} + O(R) = \pi R^2 + O(R)$$

Dirichlet's divisor problem (average number of divisors of an integer)

$$d(n) = \# \text{ positive divisors of } n = \sum_{\substack{d|n \\ d > 0}} 1 = \# \left\{ (x, y) \mid x, y \geq 1, xy = n \right\}$$

integers

Formula: (in terms of prime decomposition of  $n$ )

$$d(1) = 1, d(2) = 2, d(3) = 2, d(4) = 3$$

$$d(p) = 2 \iff p \text{ prime}, d(p^k) = k+1$$

$(p, p^2, \dots, p^k)$

Multiplicativity:

If  $m, n$  coprime  $\gcd(m, n) = 1$  then  $d(mn) = d(m)d(n)$

$$\implies n = p_1^{k_1} \dots p_r^{k_r}, p_i \text{ distinct primes}, d(n) = \prod_{j=1}^r (k_j + 1)$$

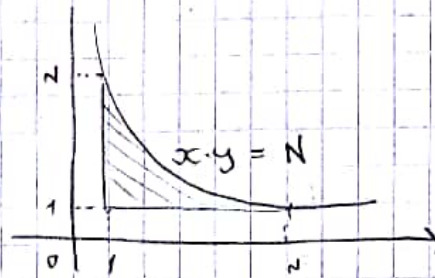
Question:  $\frac{1}{N} \sum_{n=1}^N d(n) = \frac{O(N)}{N}$  claim  $\frac{O(N)}{N} \sim \log N$

"The average number of divisors of  $n$  is  $\log n$ "

Note:

$$D(N) = \sum_{n=1}^N d(n) = \#\{(x,y) \in \mathbb{Z}^2 \mid xy \geq 1, xy \leq N\}$$

$$= \#\{(x,y) \in \mathbb{Z}^2 \mid xy \geq 1, xy = n\}$$



Thm. (Dirichlet)

$$D(N) = N \log N + (2c-1)N + \Delta(N), \text{ where } \Delta(N) = O(\sqrt{N})$$

Dirichlet divisor problem:

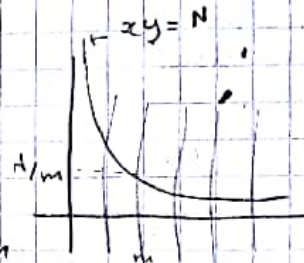
$$\forall \epsilon > 0. \Delta(N) = O(N^{1/4+\epsilon})$$

First attempt:  $D(N) \sim N \log N$

Use slicing method

$$D(N) = \sum_{1 \leq m \leq N} \lfloor N/m \rfloor = \sum_{1 \leq m \leq N} \left\{ \frac{N}{m} + O(1) \right\} =$$

$$= N \sum_{m=1}^N \frac{1}{m} + O(N) = N(\log(N) + c + o(\frac{1}{N})) = O(N) = N \log N + O(N).$$



$$\left( \sum_{m=1}^x \frac{1}{m} = \log x + c + o\left(\frac{1}{x}\right) \right)$$

Second attempt

The "hyperbola method"

$$D(N) = \square + H_1 + H_2 = \square + 2H_1$$

$$\square = \#\{(x,y) \in \mathbb{Z}^2 \mid 1 \leq x,y \leq \sqrt{N}\} = (\#\{1 \leq x \leq \sqrt{N}\})^2$$

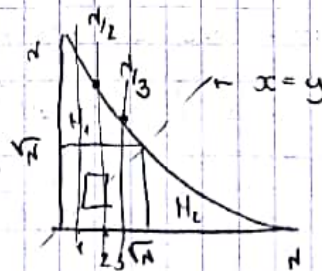
$$= (\sqrt{N} + O(1))^2 = N + O(\sqrt{N})$$

For  $H_1$  use the same slicing method.

$$H_1 = \sum_{1 \leq m \leq \sqrt{N}} \#\left\{ \sqrt{N} < k \leq \frac{N}{m} \right\} = \sum_{1 \leq m \leq \sqrt{N}} \left\{ \frac{N}{m} - \sqrt{N} \right\} + O(1) =$$

$$= N \sum_{m=1}^{\sqrt{N}} \frac{1}{m} - \sqrt{N} \lfloor \sqrt{N} \rfloor + O(\sqrt{N}) = N \left\{ \log \sqrt{N} + c + o\left(\frac{1}{\sqrt{N}}\right) \right\} - N + O(\sqrt{N})$$

$$H_1 = \frac{1}{2} N \log N + cN - N + O(\sqrt{N})$$



$$D(N) = \square + 2M, = N + O(\sqrt{N}) + 2 \left\{ \frac{1}{2} N \log N + cN - N + O(\sqrt{N}) \right\} =$$

$$= N \log N + (2c-1)N + O(\sqrt{N})$$

What we will do next time:

1906 Sierpinski, ...,  $N(R) = \pi R^2 + O(R^{2/3})$

$$\frac{1}{2} < \frac{2}{3} < 1 \quad \left( \frac{2}{3} = 0.666\dots, \text{ best known result today: } 0.6\dots \right)$$

Vander Corput = 1920: replaced  $\frac{2}{3}$  by  $\frac{2}{3} - \delta$ .

likewise for  $\Delta(R)$ .

Recall:

We argued that  $N(R) - \pi R^2$  is  $O(R^{1+\epsilon})$

by "using" independence of errors in approximation,

$$P(R) = N(R) - \pi R^2 = \sum \pm 1$$

?  $\Rightarrow \frac{P(R)}{\sqrt{R}}$  is gaussian ??

$$\frac{1}{R} \text{ meas } \{ x < R \mid x = \frac{P(x)}{\sqrt{R}} < \beta \} \xrightarrow{R \rightarrow \infty} \int_{\alpha}^{\beta} e^{-x^2} dx$$

1940's Wintner showed  $\exists$  limit

1990: Heath Brown: distribution is NON GAUSSIAN.

For next time: 1) read up on basic Fourier transform.

Poisson Summation, Fourier inversion, ...

2) Oscillatory integrals, the method of stationary phase.